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Acoustic properties of a two-fluid compressible mixture with micro-inertia

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Abstract

We study the propagation of acoustic waves in a mixture of two compressible components with micro-inertia. A special symmetric form of linearized governing equations is found permitting us to establish the stability of plane harmonic waves in the full region of flow parameters. It is shown that in non-viscous limit the dispersion relation has two real branches. One branch is defined for all disturbance frequencies, and the other one is defined only for frequencies greater than the resonance frequency. We prove that the group velocity attains its minimal value which is less than the sound speeds of each component. It is also shown that the Wood sound speed appears in our model in the limit of low frequencies only if the dissipation is taken into account. In the dissipation-free limit the equilibrium sound speed is always greater than the Wood speed.
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1. Introduction

The method we used to derive the equations describing multiphase flows is based on well-known in classical mechanics Hamilton's principle of stationary action. The main problem of this method is that the explicit form of the Lagrangian as a function of averaged parameters of the mixture is unknown. For dilute dispersed mixtures different phenomenological Lagrangians were proposed by Bedford and Drumheller [1,2], Berdichevsky [3], Geurst [4,5], Pauchon and Smereka [6] and others. In principle, if the continuous phase is incompressible and the flow around bubbles is potential, an approximate expression of the Lagrangian can be found by applying the kinetic approach to the problem of the motion of a system of bubbles [7–10]. Unfortunately, when both components (dispersed and continuous) are compressible, only the phenomenological approach to the construction of the Lagrangian can be used. For the turbulence-free case of multiphase mixture of two compressible components with micro-inertia effects this Lagrangian was proposed in Gavrilyuk and Saurel [11] and for the turbulence case in Saurel, Gavrilyuk and Renaud [12]. To derive the governing equations, we applied the technique developed before in Gavrilyuk and Gouin [13] which does not use the conventional method of Lagrange multiplies. The model obtained is hyperbolic that permits us to study the sound wave propagation. We give below a brief description of this model.

Each a-th component ($a = 1, 2$) has its own averaged characteristics: the local velocity $\mathbf{u}_a$; the local (true) densities ρ_a; the partial densities $\bar{\rho}_a = \alpha_a \rho_a$; α_a is the volume fraction of the a-th component, $\alpha_1 + \alpha_2 = 1$ (if there is no ambiguity, we will

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denote α_2 by α); the local entropies η_a, the local internal energies per unit mass $e_a(\rho_a,\eta_a)$, and the local temperatures θ_a. The energies $e_a(\rho_a,\eta_a)$ verify the Gibbs identity:

$$\theta_a\,\mathrm{d}\eta_a=\mathrm{d}e_a+p_a\,\mathrm{d}\left(\frac{1}{\rho_a}\right),$$

where p_a are the pressures. We introduce also the energies related with inertia effects (pulsations of components). The expression of these energies in terms of the averaged mixture parameters is under discussion [14]. We used the volume fraction variables for the micro-inertia description. The pulsation energies per unit volume in each phase are taken in the form

$$\frac{M_a}{2}\left(\frac{\mathrm{d}_a\alpha_a}{\mathrm{d}t}\right)^2,\qquad \frac{\mathrm{d}_a}{\mathrm{d}t}=\frac{\partial}{\partial t}+\mathbf{u}_a\cdot\nabla,$$

where M_a are given functions of flow parameters. Let the index "2" denotes the dispersed phase ("bubbles"). The contribution of the pulsation energy is important when the compressibility of the dispersed phase is much more higher than that of the continuous phase. In this case the pulsation energy in the dispersed phase very often can be negligible ($M_2=0$). In the following we suppose that

$$M_1=M(\alpha,\bar\rho_1).$$

For example, for a bubbly fluid with incompressible liquid phase ($\rho_1=const$) this energy can be written as $2\pi R^3\rho_1 N(\frac{\mathrm{d}R}{\mathrm{d}t})^2$, where R is the averaged bubble radius, and N is the number of bubbles per unit volume. The volume concentration is given by the formula: $\alpha_2=\alpha=\frac{4}{3}\pi R^3 N$. Then the function M can be taken as $M=m(\alpha)$ as was done in [11]. If the liquid phase is slightly compressible, a better approximation is $M=\bar\rho_1 m_1(\alpha)$. Both approximations are almost equivalent.

A simplified turbulence-free dissipative model for $M=\bar\rho_1 m_1(\alpha)=\alpha_1\rho_1 m_1(\alpha)$ is (see [12]):

$$\begin{aligned}
&\frac{\partial\alpha_1\rho_1}{\partial t}+\operatorname{div}(\alpha_1\rho_1\mathbf{u}_1)=0,\\
&\frac{\partial\alpha_2\rho_2}{\partial t}+\operatorname{div}(\alpha_2\rho_2\mathbf{u}_2)=0,\\
&\frac{\partial\alpha_1\rho_1\mathbf{u}_1}{\partial t}+\operatorname{div}(\alpha_1\rho_1\mathbf{u}_1\otimes\mathbf{u}_1+\alpha_1p_1I)=p_2\nabla\alpha_1+\lambda(\mathbf{u}_2-\mathbf{u}_1),\\
&\frac{\partial\alpha_2\rho_2\mathbf{u}_2}{\partial t}+\operatorname{div}(\alpha_2\rho_2\mathbf{u}_2\otimes\mathbf{u}_2+\alpha_2p_2I)=p_2\nabla\alpha_2-\lambda(\mathbf{u}_2-\mathbf{u}_1),\\
&\frac{\partial}{\partial t}\left(\alpha_1\rho_1\left(\frac{|\mathbf{u}_1|^2}{2}+e_1+\frac{\tau^2}{2}\right)\right)+\operatorname{div}\left(\alpha_1\rho_1\mathbf{u}_1\left(\frac{|\mathbf{u}_1|^2}{2}+e_1+\frac{\tau^2}{2}\right)+\alpha_1\mathbf{u}_1p_1\right)=-p_2\frac{\partial\alpha_1}{\partial t}+\lambda\mathbf{u}_1\cdot(\mathbf{u}_2-\mathbf{u}_1),\\
&\frac{\partial}{\partial t}\left(\alpha_2\rho_2\left(\frac{|\mathbf{u}_2|^2}{2}+e_2\right)\right)+\operatorname{div}\left(\alpha_2\rho_2\mathbf{u}_2\left(\frac{|\mathbf{u}_2|^2}{2}+e_2\right)+\alpha_2\mathbf{u}_2p_2\right)=-p_2\frac{\partial\alpha_2}{\partial t}-\lambda\mathbf{u}_1\cdot(\mathbf{u}_2-\mathbf{u}_1),\\
&\frac{\mathrm{d}_1\tau}{\mathrm{d}t}=\frac{p_2-p_1-\chi\tau}{\alpha_1\rho_1\sqrt{m_1}},\qquad \frac{\mathrm{d}_1\alpha}{\mathrm{d}t}=\frac{\tau}{\sqrt{m_1}}.
\end{aligned}\tag{1.1}$$

The following remarks can be done:

- The non-dissipative part of system (1.1) (i.e. $\lambda=0,\ \chi=0$) corresponds to the Lagrangian [11,12]:

  $$L=\sum_{a=1}^{2}\bar\rho_a\left(\frac{|\mathbf{u}_a|^2}{2}-e_a(\rho_a,\eta_a)\right)+\frac{\bar\rho_1m_1(\alpha)}{2}\left(\frac{\mathrm{d}_1\alpha}{\mathrm{d}t}\right)^2.$$

  The added mass term was negligible for our applications.
- The last two equations of (1.1) generalize the Rayleigh–Lamb equation (see, for example, Drew and Passman [15]).
- The term $\lambda(\mathbf{u}_2-\mathbf{u}_1)$ is the Stokes type friction force, λ is a positive scalar function depending on the local parameters of the mixture and the relative velocity of components.
- The term $\chi\tau$ is a friction force responsible for the pulsation damping, function χ is positive.
- If we neglect the pulsation velocity τ in the energy equation and microacceleration $\mathrm{d}_1\tau/\mathrm{d}t$ in the generalized Rayleigh–Lamb equation, we obtain the Baer–Nunziato equations for non-reacting immiscible mixtures, or the BN model (Baer and Nunziato [16]) as a particular case of model (1.1) (see also [12]):

$$\begin{aligned}
&\frac{\partial\alpha_1\rho_1}{\partial t}+\operatorname{div}(\alpha_1\rho_1\mathbf{u}_1)=0,\\
&\frac{\partial\alpha_2\rho_2}{\partial t}+\operatorname{div}(\alpha_2\rho_2\mathbf{u}_2)=0,\\
&\frac{\partial\alpha_1\rho_1\mathbf{u}_1}{\partial t}+\operatorname{div}(\alpha_1\rho_1\mathbf{u}_1\otimes\mathbf{u}_1+\alpha_1p_1I)=p_2\nabla\alpha_1+\lambda(\mathbf{u}_2-\mathbf{u}_1),\\
&\frac{\partial\alpha_2\rho_2\mathbf{u}_2}{\partial t}+\operatorname{div}(\alpha_2\rho_2\mathbf{u}_2\otimes\mathbf{u}_2+\alpha_2p_2I)=p_2\nabla\alpha_2-\lambda(\mathbf{u}_2-\mathbf{u}_1),\\
&\frac{\partial}{\partial t}\left(\alpha_1\rho_1\left(\frac{|\mathbf{u}_1|^2}{2}+e_1\right)\right)+\operatorname{div}\left(\alpha_1\rho_1\mathbf{u}_1\left(\frac{|\mathbf{u}_1|^2}{2}+e_1\right)+\alpha_1\mathbf{u}_1p_1\right)=-p_2\frac{\partial\alpha_1}{\partial t}+\lambda\mathbf{u}_1\cdot(\mathbf{u}_2-\mathbf{u}_1),\\
&\frac{\partial}{\partial t}\left(\alpha_2\rho_2\left(\frac{|\mathbf{u}_2|^2}{2}+e_2\right)\right)+\operatorname{div}\left(\alpha_2\rho_2\mathbf{u}_2\left(\frac{|\mathbf{u}_2|^2}{2}+e_2\right)+\alpha_2\mathbf{u}_2p_2\right)=-p_2\frac{\partial\alpha_2}{\partial t}-\lambda\mathbf{u}_1\cdot(\mathbf{u}_2-\mathbf{u}_1),\\
&\frac{\mathrm{d}_1\alpha}{\mathrm{d}t}=\mu(p_2-p_1),\qquad \mu=\frac{1}{\chi\sqrt{m_1}}.
\end{aligned}\tag{1.2}$$

Models (1.1) and (1.2) correspond to a particular choice of the interfacial pressure P_I and the interfacial velocity $\mathbf{u}_I : P_I = p_2$ (the interfacial pressure is that of the dispersed phase) and $\mathbf{u}_I=\mathbf{u}_1$ (the interfacial velocity is that of the continuous phase). Formally, other choice of these characteristics compatible with the second law of thermodynamics is possible (see, for example, Gallouët et al. [17]). The symmetric form of governing equations we present below does not depend on a particular choice of interfacial variables. Model (1.2) was used for a two-phase modeling of deflagration-to-detonation transition in granular materials (Bdzil et al. [18], Kapila et al. [19,20]). Glimm et al. [21] studied a Rayleigh–Taylor mixing layer by using also the BN type two-fluid model. Models (1.1) and (1.2) were used as 1D reduced models in studying phenomena that is essentially multi-dimensional: shock wave interaction with a big bubble [11,12]. A good agreement between numerical results by the reduced 1D model and averaged results of the 2D Euler equations has been obtained. Model (1.1) was also tested on the problem of shock wave propagation in bubbly fluids. A very good agreement between experimental and numerical data has been found. Here we study the propagation of linear waves for system (1.1) and compare the results obtained with model (1.2). In Section 2 we present a special symmetric form of governing equations permitting to prove the stability of plane harmonic waves without hard calculations. We analyze in detail in Section 3 the dissipation-free dispersion relation. In Section 4 the dissipative dispersion relation is studied. In particular, we find the Wood sound velocity in the low-frequency limit from the dissipative dispersion relation.

2. Symmetrization of linearized equations and stability analysis

We shall study the linear stability of plane harmonic waves. Usually, this routine procedure gives us a polynomial dispersion relation which cannot be solved analytically. Only the asymptotic analysis of the dispersion relation is possible (small viscosities, low or high frequencies, etc.) However, we find here an interesting symmetric form of governing equations permitting us to prove the stability in the full region of flow parameters. We think that this form may be useful for numerical applications.

We consider perturbations of an equilibrium constant state denoted by index "0":

$$\begin{aligned}
&\bar\rho_a=\bar\rho_{a0}+\varepsilon\bar\rho_a',\quad \mathbf{u}_a=\mathbf{u}_0+\varepsilon\mathbf{u}_a',\quad p_a=p_0+\varepsilon p_a',\\
&\alpha_a=\alpha_{a0}+\varepsilon\alpha_a',\quad \tau=\varepsilon\tau',\quad \eta_a=\eta_{a0}+\varepsilon\eta_a'.
\end{aligned}$$

Since the governing equations are Galilean invariant, we can always suppose that $\mathbf{u}_0=0$. The linearized equations are:

$$\begin{aligned}
&\frac{\partial\bar\rho_2'}{\partial t}+\bar\rho_{20}\operatorname{div}\mathbf{u}_2'=0,\\
&\frac{\partial\bar\rho_1'}{\partial t}+\bar\rho_{10}\operatorname{div}\mathbf{u}_1'=0,\\
&\bar\rho_{20}\frac{\partial\mathbf{u}_2'}{\partial t}+\alpha_{20}\nabla p_2'=\lambda_0(\mathbf{u}_1'-\mathbf{u}_2'),\\
&\bar\rho_{10}\frac{\partial\mathbf{u}_1'}{\partial t}+\alpha_{10}\nabla p_1'=-\lambda_0(\mathbf{u}_1'-\mathbf{u}_2'),\\
&\frac{\partial\alpha'}{\partial t}=\frac{\tau'}{\sqrt{m_{10}}},
\end{aligned}\tag{2.3}$$

$$\frac{\partial \tau'}{\partial t} = \frac{1}{\bar\rho_{10}\sqrt{m_{10}}}(p'_2 - p'_1 - \chi_0\tau'),$$
$$\frac{\partial \eta'_1}{\partial t} = 0, \qquad \frac{\partial \eta'_2}{\partial t} = 0.$$

Here $\lambda_{0,}$ m_{10} and χ_0 are equilibrium positive constants. We suppose that the flow is isentropic, i.e. η'_1 et η'_2 are constant (the equations for entropies are equivalent to the energy equations). In the isentropic case

$$p'_2 = c_2^2\left(\frac{\bar\rho'_2}{\alpha_{20}} - \frac{\bar\rho_{20}}{\alpha_{20}^2}\alpha'\right), \qquad p'_1 = c_1^2\left(\frac{\bar\rho'_1}{\alpha_{10}} + \frac{\bar\rho_{10}}{\alpha_{10}^2}\alpha'\right),$$

where c_a are the sound velocities. This yields

$$\bar\rho'_2 = \frac{\alpha_{20}}{c_2^2}p'_2 + \frac{\bar\rho_{20}}{\alpha_{20}}\alpha', \qquad \bar\rho'_1 = \frac{\alpha_{10}}{c_1^2}p'_1 - \frac{\bar\rho_{10}}{\alpha_{10}}\alpha'.$$

Replacing these expressions in the mass conservation equations and multiplying them by $\alpha_{20}/\bar\rho_{20}$ and $\alpha_{10}/\bar\rho_{10}$, respectively, we obtain:

$$\frac{\alpha_{20}^2}{\bar\rho_{20}c_2^2}\frac{\partial p'_2}{\partial t} + \alpha_{20}\,\mathrm{div}\,\mathbf{u}'_2 + \frac{\partial\alpha'}{\partial t} = 0,$$
$$\frac{\alpha_{10}^2}{\bar\rho_{10}c_1^2}\frac{\partial p'_1}{\partial t} + \alpha_{10}\,\mathrm{div}\,\mathbf{u}'_1 - \frac{\partial\alpha'}{\partial t} = 0,$$
$$\bar\rho_{20}\frac{\partial \mathbf{u}'_2}{\partial t} + \alpha_{20}\nabla p'_2 = \lambda_0(\mathbf{u}'_1 - \mathbf{u}'_2),$$
$$\bar\rho_{10}\frac{\partial \mathbf{u}'_1}{\partial t} + \alpha_{10}\nabla p'_1 = -\lambda_0(\mathbf{u}'_1 - \mathbf{u}'_2),$$
$$\frac{\partial\alpha'}{\partial t} = \frac{\tau'}{\sqrt{m_{10}}},$$
$$\frac{\partial \tau'}{\partial t} = \frac{1}{\bar\rho_{10}\sqrt{m_{10}}}(p'_2 - p'_1 - \chi_0\tau').$$

Replacing the derivative $\partial\alpha'/\partial t$ by its algebraic expression and multiplying the equation for τ' by $\bar\rho_{10}$, we get finally:

$$\begin{aligned}
&\frac{\alpha_{20}^2}{\bar\rho_{20}c_2^2}\frac{\partial p'_2}{\partial t} + \alpha_{20}\,\mathrm{div}\,\mathbf{u}'_2 = -\frac{\tau'}{\sqrt{m_{10}}},\\
&\frac{\alpha_{10}^2}{\bar\rho_{10}c_1^2}\frac{\partial p'_1}{\partial t} + \alpha_{10}\,\mathrm{div}\,\mathbf{u}'_1 = \frac{\tau'}{\sqrt{m_{10}}},\\
&\bar\rho_{20}\frac{\partial \mathbf{u}'_2}{\partial t} + \alpha_{20}\nabla p'_2 = \lambda_0(\mathbf{u}'_1 - \mathbf{u}'_2),\\
&\bar\rho_{10}\frac{\partial \mathbf{u}'_1}{\partial t} + \alpha_{10}\nabla p'_1 = -\lambda_0(\mathbf{u}'_1 - \mathbf{u}'_2),\\
&\bar\rho_{10}\frac{\partial \tau'}{\partial t} = \frac{1}{\sqrt{m_{10}}}(p'_2 - p'_1 - \chi_0\tau'),\\
&\frac{\partial\alpha'}{\partial t} = \frac{\tau'}{\sqrt{m_{10}}}.
\end{aligned} \tag{2.4}$$

The first five equations of (2.4) are completely independent and can be rewritten in symmetric form:

$$A\frac{\partial \mathbf{U}}{\partial t} + \sum B^j\frac{\partial \mathbf{U}}{\partial x_j} = C_s\mathbf{U} + C_a\mathbf{U} \tag{2.5}$$

with

$$\mathbf{U} = (p'_2,\ p'_1, \mathbf{u}'_2, \mathbf{u}'_1, \tau')^{\mathrm{T}}, \quad A = A^{\mathrm{T}} > 0, \quad B^i = (B^i)^{\mathrm{T}}, \quad C_s = C_s^{\mathrm{T}} \leqslant 0, \quad C_a = -C_a^{\mathrm{T}}.$$

Hence, the equations for $\mathbf{U}$ et α' are divided into two subsystems: equations for $\mathbf{U}$ which are symmetric t-hyperbolic and dissipative ($C_s \leqslant 0$), and the equation for α'. In dissipation-free case the matrix C_s is zero. This symmetric form generalizes the well-known symmetric form of the gas dynamics equations in variables "velocity-pressure-entropy" (for simplicity, we give here only the isentropic linearized version):

$$\frac{1}{\rho c^2}\frac{\partial p'}{\partial t} + \operatorname{div}\mathbf{u}' = 0,$$
$$\rho\frac{\partial \mathbf{u}'}{\partial t} + \nabla p' = 0.$$

If the perturbations $\mathbf{U}$ are localized in space, we immediately obtain from (2.5) the stability estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathbf{U}^{\mathrm{T}} A\mathbf{U}\,\mathrm{d}\mathbf{x} = \int \mathbf{U}^{\mathrm{T}} C_s\mathbf{U}\,\mathrm{d}\mathbf{x} \leqslant 0.$$

The linearized BN model can be transformed in the same way:

$$\begin{aligned}
&\frac{\alpha_{20}^2}{\bar\rho_{20}c_2^2}\frac{\partial p_2'}{\partial t} + \alpha_{20}\operatorname{div}\mathbf{u}_2' = -\mu_0(p_2' - p_1'),\\
&\frac{\alpha_{10}^2}{\bar\rho_{10}c_1^2}\frac{\partial p_1'}{\partial t} + \alpha_{10}\operatorname{div}\mathbf{u}_1' = \mu_0(p_2' - p_1'),\\
&\bar\rho_{20}\frac{\partial \mathbf{u}_2'}{\partial t} + \alpha_{20}\nabla p_2' = \lambda_0(\mathbf{u}_1' - \mathbf{u}_2'),\\
&\bar\rho_{10}\frac{\partial \mathbf{u}_1'}{\partial t} + \alpha_{10}\nabla p_1' = -\lambda_0(\mathbf{u}_1' - \mathbf{u}_2'),\\
&\frac{\partial \alpha'}{\partial t} = \mu_0(p_2' - p_1').
\end{aligned} \tag{2.6}$$

So, the first four independent equations can also be rewritten in symmetric form and the last equation permits us to find α'.

We are looking for the solution of linearized equations of the form:

$$\mathbf{U} = \mathcal{U}\,\mathrm{e}^{\mathrm{i}(\mathbf{k}^{\mathrm{T}}\mathbf{x} - \omega t)}, \quad \mathbf{k} = (k_1, k_2, k_3)^{\mathrm{T}},$$

where $\mathbf{k}$ is the *wave vector* and ω is the *frequency*. Replacing in (2.5) we obtain:

$$\Big(A(-\mathrm{i}\omega) + \mathrm{i}\sum B^j k_j - C_s - C_a\Big)\mathcal{U} = 0.$$

Hence, the dispersion relation $\omega(\mathbf{k})$ is a solution of the algebraic equation

$$\det\Big(A(-\mathrm{i}\omega) + \mathrm{i}\sum B^j k_j - C_s - C_a\Big) = 0. \tag{2.7}$$

For real $\mathbf{k}$ we define the *phase velocity* (which is complex a priori) $c_p = \omega/|\mathbf{k}|$. We will show that $\Im(c_p) = 0$, if $C_s = 0$, and $\Im(c_p) \leqslant 0$, if $C_s \leqslant 0$. It means that the plane harmonic waves are stable.

Indeed, let $c_p = \Re(c_p) + \mathrm{i}\Im(c_p)$. Then

$$\Big(\sum B^j n_j + \mathrm{i}\frac{C_s}{|\mathbf{k}|} + \mathrm{i}\frac{C_a}{|\mathbf{k}|} - c_p A\Big)\mathcal{U} = 0.$$

Here $n_j = k_j/|\mathbf{k}|$. Multiplying this expression from the left by $\overline{\mathcal{U}}^{\mathrm{T}}$ where "bar" means complex conjugate, we get:

$$\overline{\mathcal{U}}^{\mathrm{T}}\Big(\sum B^j n_j + \mathrm{i}\frac{C_s}{|\mathbf{k}|} + \mathrm{i}\frac{C_a}{|\mathbf{k}|} - c_p A\Big)\mathcal{U} = 0.$$

It is equivalent to

$$\overline{\mathcal{U}}^{\mathrm{T}}\Big(\sum B^j n_j + \mathrm{i}\frac{C_a}{|\mathbf{k}|}\Big)\mathcal{U} + \frac{\mathrm{i}}{|\mathbf{k}|}\overline{\mathcal{U}}^{\mathrm{T}} C_s\mathcal{U} - c_p\overline{\mathcal{U}}^{\mathrm{T}} A\mathcal{U} = 0.$$

Since the first term is real, we obtain the following expression for $\Im(c_p)$:

$$\Im(c_p) = \frac{1}{|\mathbf{k}|}\frac{\overline{\mathcal{U}}^{\mathrm{T}} C_s\mathcal{U}}{\overline{\mathcal{U}}^{\mathrm{T}} A\mathcal{U}} \leqslant 0.$$

The stability of plane harmonic waves is proved. It is worth to note that other possible proofs of stability are related with enormous calculations. For example, the direct stability analysis of non-symmetric system (2.3) by using the Routh–Hurwitz criterion takes about 16 pages [22].

3. Dispersion relation for the dissipation-free case ($C_s = 0$)

Now we shall study the dispersion relation for model (2.4) in detail. Since the governing equations are invariant under rotation, it is sufficient to consider the 1D case. We drop the subscript "0" in dispersion relation (2.7):

$$\det(-\mathrm{i}\omega A + \mathrm{i}kB - C_a) = 0$$

with the matrices $A,\ B,\ C_a$ defined by

$$A = \begin{pmatrix} \frac{\alpha_2^2}{\bar\rho_2 c_2^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha_1^2}{\bar\rho_1 c_1^2} & 0 & 0 & 0 \\ 0 & 0 & \bar\rho_2 & 0 & 0 \\ 0 & 0 & 0 & \bar\rho_1 & 0 \\ 0 & 0 & 0 & 0 & \bar\rho_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_a = \begin{pmatrix} 0 & 0 & 0 & 0 & -1/\sqrt{m_1} \\ 0 & 0 & 0 & 0 & 1/\sqrt{m_1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1/\sqrt{m_1} & -1/\sqrt{m_1} & 0 & 0 & 0 \end{pmatrix}.$$

The dispersion relation has five roots, the first is $\omega = 0$ and four other roots are solutions of the equation

$$M\omega^4 - \left(\frac{\bar\rho_1 c_1^2}{\alpha_1^2} + \frac{\bar\rho_2 c_2^2}{\alpha_2^2} + Mk^2\big(c_1^2 + c_2^2\big)\right)\omega^2 + c_1^2 c_2^2 k^2\left(Mk^2 + \frac{\bar\rho_1}{\alpha_1^2} + \frac{\bar\rho_2}{\alpha_2^2}\right) = 0.$$

Here $M = \bar\rho_1 m_1$. It is more convenient to rewrite this dispersion relation in terms of the *inverse phase velocity* $c_p^{-1} = k/\omega$ and the frequency ω:

$$\frac{1}{c_p^4} - \frac{1}{c_p^2}\left(c_1^{-2} + c_2^{-2} - \frac{\bar\rho_1/\alpha_1^2 + \bar\rho_2/\alpha_2^2}{M\omega^2}\right) + \frac{1}{c_1^2 c_2^2}\left(1 - \frac{\bar\rho_1 c_1^2/\alpha_1^2 + \bar\rho_2 c_2^2/\alpha_2^2}{M\omega^2}\right) = 0. \tag{3.8}$$

We introduce the *resonance frequency* ω_0 corresponding to the case when one of the roots $1/c_p^2$ vanishes:

$$\omega_0^2 = \frac{\bar\rho_1 c_1^2/\alpha_1^2 + \bar\rho_2 c_2^2/\alpha_2^2}{M} = \frac{\rho_1 c_1^2/\alpha_1 + \rho_2 c_2^2/\alpha_2}{M} \tag{3.9}$$

and the *equilibrium sound speed* c_e corresponding to the limit $\omega \to 0$

$$c_e^2 = \frac{(\rho_1/\alpha_1 + \rho_2/\alpha_2)c_1^2 c_2^2}{\rho_1 c_1^2/\alpha_1 + \rho_2 c_2^2/\alpha_2}. \tag{3.10}$$

We suppose that $c_1 > c_2$. Then it follows from (3.10) that

$$c_1 > c_e > c_2.$$

With these definitions, the equation for $Z = 1/c_p^2$ can be rewritten as

$$Z^2 - \frac{1}{c_1^2 c_2^2}\left(c_1^2 + c_2^2 - c_e^2\frac{\omega_0^2}{\omega^2}\right)Z + \frac{1}{c_1^2 c_2^2}\left(1 - \frac{\omega_0^2}{\omega^2}\right) = 0. \tag{3.11}$$

A simple analysis of (3.11) shows that

- There is only one positive root $Z_2(\omega)$ for $\omega < \omega_0$, and two positive roots $Z_2(\omega) > Z_1(\omega)$ for $\omega > \omega_0$, $Z_1(0) = 0$;
- $1/c_e^2 < Z_2(\omega) < 1/c_2^2$ for $0 < \omega < \infty$, $0 < Z_1(\omega) < 1/c_1^2$ for $\omega_0 < \omega < \infty$;

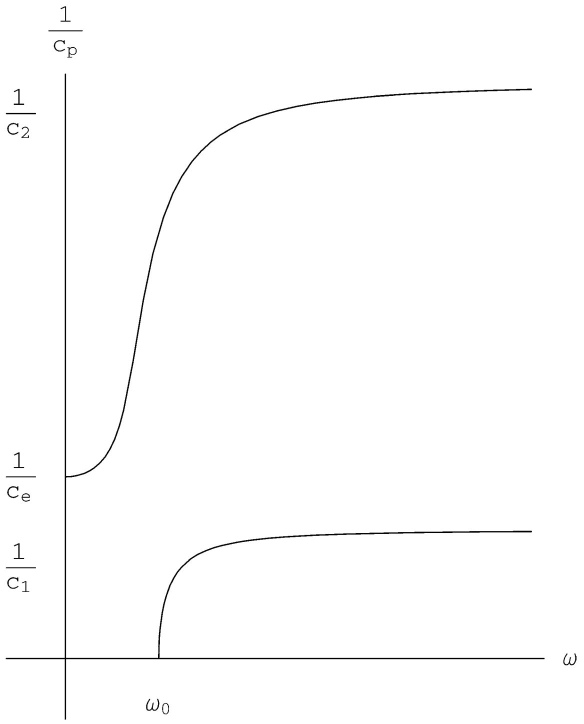

Fig. 1. The dispersion curve $c_p^{-1}(\omega)$ has two branches. The first branch is defined only on the interval (ω_0, ∞) and is monotonic and concave function of ω. The second branch is defined on the whole interval $(0, \infty)$. It is monotonic function of ω and has a unique inflection point.

- $Z_2(0) = 1/c_e^2$, $\lim_{\omega\to\infty} Z_2(\omega) = 1/c_2^2$, $\lim_{\omega\to\infty} Z_1(\omega) = 1/c_1^2$;
- $Z_i\,(\omega)$ are increasing functions of ω, $i = 1, 2$.

The inverse phase velocity as a function of frequency is shown in Fig. 1. A striking similarity between the results obtained and those by Twiss and Eringen [23] for a binary mixture of isotropic micropolar elastic materials can be observed (see also Bedford and Drumheller [2, p. 935]). However, the behavior of the *group velocity* is different.

The group velocity is defined by the relation

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

This definition implies

$$\frac{1}{c_g} = \frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{k}{\omega}\omega\right) = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\omega}{c}\right) = \frac{1}{c} + \omega\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{c}\right).$$

This formula is useful to study the behavior of $1/c_g$ as a function of ω. It can be shown that

- The inverse group velocity corresponding to the first branch $Z_1(\omega)$, defined on the interval (ω_0, ∞), is decreasing function of ω. Furthermore,

$$\lim_{\omega\to\omega_0+0}\frac{1}{c_g} = \infty, \qquad \lim_{\omega\to\infty}\frac{1}{c_g} = \frac{1}{c_1};$$

- The inverse group velocity, corresponding to the second branch $Z_2(\omega)$, is defined on the interval $(0, \infty)$ (Fig. 2). Furthermore,

$$\lim_{\omega\to 0}\frac{1}{c_g} = \frac{1}{c_e}, \qquad \lim_{\omega\to\infty}\frac{1}{c_g} = \frac{1}{c_2},$$

  and there is the frequency ω_c: $0 < \omega_c < \omega_0$ such that $1/c_g$ has a local maximum at ω_c (so, the group velocity has a minimal value). The group velocity at this point is less then both c_e and c_2. The last statement is very important because the group

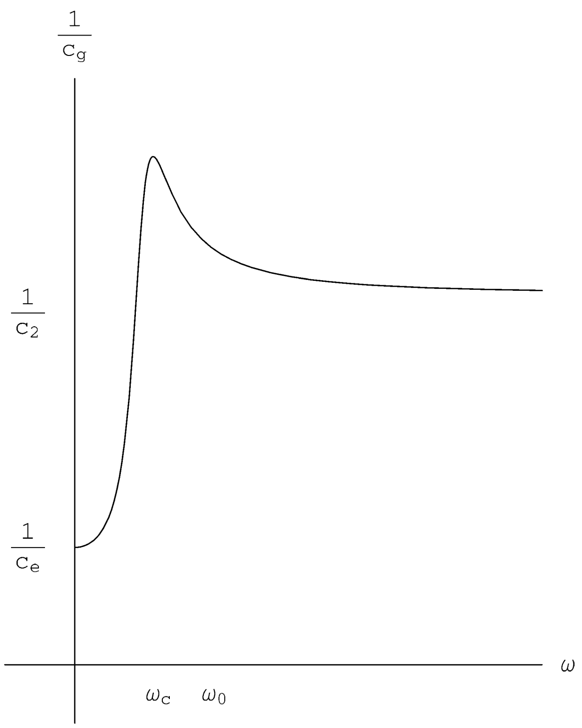

Fig. 2. The inverse group velocity graph is shown for the second branch defined on the whole interval $(0, \infty)$. At point $\omega_c < \omega_0$ it attains its maximal value. It corresponds to the group velocity having the value less than c_2 and c_e.

velocity is responsible for the energy translation. So, the energy can propagate with the velocity which is slower then the sound speeds of each phase.

The dissipation-free model of Baer–Nunziato (2.6) is much more simpler. The dispersion curves are just two straight lines $c_p = c_1$ and $c_p = c_2$.

We note that for the one-velocity bubbly fluids the dissipation-free dispersion relation has an interval of frequencies where there is no sonic waves propagating through the medium (“non-transparency” interval) (see, for example, Nigmatulin [24]). Fig. 1 shows that the dispersion relation obtained for the two-velocity model has no such an interval. There always exists at least one sonic wave if the disturbance frequency is less then the resonance frequency, and two waves if the disturbance frequency is higher than the resonance frequency.

What is surprising in the analysis we have done is the fact that the famous Wood sound speed which is considered by all specialists in the multiphase flow theory as a true low-frequency sound speed, does not appear in this dissipation-free limit. We recall that this sound speed is given by [25, p. 327]

$$c_W^2 = \frac{1}{(\bar\rho_1 + \bar\rho_2)(\alpha_2^2/(\bar\rho_2 c_2^2) + \alpha_1^2/(\bar\rho_1 c_1^2))} = \frac{1}{(\alpha_1\rho_1 + \alpha_2\rho_2)(\alpha_2/(\rho_2 c_2^2) + \alpha_1/(\rho_1 c_1^2))}. \tag{3.12}$$

It can be shown that

$$c_e^2 > c_W^2.$$

Since many experiments show that the Wood formula is quite correct (see, for example, the book by Drew and Passman [15]), it is interesting to understand what is the reason of this discrepancy.

4. Wood's sound speed

We consider now the dissipative case ($C_s \neq 0$). The analysis for the BN-model is simpler and we give it in detail. Since $C_a = 0$ and

$$A = \begin{pmatrix} \frac{\alpha_2^2}{\bar\rho_2 c_2^2} & 0 & 0 & 0 \\ 0 & \frac{\alpha_1^2}{\bar\rho_1 c_1^2} & 0 & 0 \\ 0 & 0 & \bar\rho_2 & 0 \\ 0 & 0 & 0 & \bar\rho_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_1 \\ \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \end{pmatrix}, \quad C_s = \begin{pmatrix} -\mu & \mu & 0 & 0 \\ \mu & -\mu & 0 & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & \lambda & -\lambda \end{pmatrix},$$

the dispersion relation for the BN-model is determined by the equation:

$$\det\begin{pmatrix} -\mathrm{i}\omega\frac{\alpha_2^2}{\bar\rho_2 c_2^2} + \mu & -\mu & \mathrm{i}k\alpha_2 & 0 \\ -\mu & -\mathrm{i}\omega\frac{\alpha_1^2}{\bar\rho_1 c_1^2} + \mu & 0 & \mathrm{i}k\alpha_1 \\ \mathrm{i}k\alpha_2 & 0 & -\mathrm{i}\omega\bar\rho_2 + \lambda & -\lambda \\ 0 & \mathrm{i}k\alpha_1 & -\lambda & -\mathrm{i}\omega\bar\rho_1 + \lambda \end{pmatrix} = 0.$$

This is a polynomial of the fourth degree with respect to both the frequency and the wave number:

$$\begin{aligned} &\mathrm{i}\omega^4\bar\rho_1\bar\rho_2 + \omega^3\left(-\lambda(\bar\rho_1+\bar\rho_2) - \mu\bar\rho_1\bar\rho_2\left(\frac{\bar\rho_1 c_1^2}{\alpha_1^2} + \frac{\bar\rho_2 c_2^2}{\alpha_2^2}\right)\right) + \mathrm{i}\omega^2\left(-\lambda\mu(\bar\rho_1+\bar\rho_2)\left(\frac{\bar\rho_1 c_1^2}{\alpha_1^2} + \frac{\bar\rho_2 c_2^2}{\alpha_2^2}\right) - \bar\rho_1\bar\rho_2 k^2(c_1^2+c_2^2)\right) \\ &\quad + \omega\left(\mu\bar\rho_1\bar\rho_2 k^2 c_1^2 c_2^2\left(\frac{\bar\rho_1}{\alpha_1^2} + \frac{\bar\rho_2}{\alpha_2^2}\right) + \lambda k^2(\bar\rho_1 c_1^2 + \bar\rho_2 c_2^2)\right) + \mathrm{i}\left(\bar\rho_1\bar\rho_2 k^2 c_1^2 c_2^2\left(k^2 + \frac{\lambda\mu}{\alpha_1^2\alpha_2^2}\right)\right) = 0. \end{aligned}$$

Supposing that λ and μ are finite, we get in the limit $k \to 0$, $\omega \to 0$ the Wood sound velocity:

$$\frac{\omega^2}{k^2} = \frac{1}{(\bar\rho_1+\bar\rho_2)(\alpha_2^2/(\bar\rho_2 c_2^2) + \alpha_1^2/(\bar\rho_1 c_1^2))} = c_W^2.$$

Now we consider model (2.4) in the dissipative case. The matrix C_s has the following form:

$$C_s = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & \lambda & 0 \\ 0 & 0 & \lambda & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\frac{\chi}{\sqrt{m_1}} \end{pmatrix}.$$

The dispersion relation is:

$$\det\begin{pmatrix} -\mathrm{i}\omega\frac{\alpha_2^2}{\bar\rho_2 c_2^2} & 0 & \mathrm{i}k\alpha_2 & 0 & \frac{1}{\sqrt{m_1}} \\ 0 & -\mathrm{i}\omega\frac{\alpha_1^2}{\bar\rho_1 c_1^2} & 0 & \mathrm{i}k\alpha_1 & -\frac{1}{\sqrt{m_1}} \\ \mathrm{i}k\alpha_2 & 0 & -\mathrm{i}\omega\bar\rho_2 + \lambda & -\lambda & 0 \\ 0 & \mathrm{i}k\alpha_1 & -\lambda & -\mathrm{i}\omega\bar\rho_1 + \lambda & 0 \\ -\frac{1}{\sqrt{m_1}} & \frac{1}{\sqrt{m_1}} & 0 & 0 & -\mathrm{i}\omega\bar\rho_1 + \frac{\chi}{\sqrt{m_1}} \end{pmatrix} = 0.$$

The developed form of this determinant is a polynomial of fifth degree with respect to ω and forth degree with respect to k. Supposing that the relaxation coefficients λ and χ are finite, we find in the same manner the Wood sound speed as the velocity of low-frequency waves.

So, for the models we presented here, the Wood sound speed corresponds not to the phase velocity of low-frequency waves in the dissipation-free model, but to the phase velocity of low-frequency waves in the dissipative model.

5. Conclusion

- We have studied the stability of plane harmonic waves for a hyperbolic multiphase model describing the motion of two compressible components with micro-inertia. The linearized model has been rewritten in a special symmetric form permitting us to prove the linear stability without hard calculations.

- It has been shown that in the dissipation-free limit two sound waves exist if the disturbance frequency is greater than the resonance frequency, and only one, if the disturbance frequency is less than the resonance frequency. The group velocity attains its minimum which is less than the sound speeds in each phase. This fact is very important because the energy of sonic waves propagates with the group velocity.
- It has been shown that the Wood sound speed corresponds to the low-frequency limit in the dissipative model.

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